Commutative Algebra
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1 More on Krull Dimension

Throughout this section we will assume that R is a commutative ring.

Proposition 1 *Let R be Artinian, then $K-\dim R = 0$ (i.e., every prime ideal is maximal).*

Proof. Let P be a prime ideal of R. ${}^{R}\!/\!{}_{P}$ is an integral domain and is Artinian. Take $a \neq 0, a \in {}^{R}\!/\!{}_{P}$, then $({}^{R}\!/\!{}_{P})\, a \supseteq ({}^{R}\!/\!{}_{P})\, a^2 \supseteq \ldots$ is a descending chain, so $({}^{R}\!/\!{}_{P})\, a^n = ({}^{R}\!/\!{}_{P})\, a^{n+1}$, so $a^n = ba^{n+1}$ for some $b \in {}^{R}\!/\!{}_{P}$, so $1 = ba$, and a is a unit. Therefore ${}^{R}\!/\!{}_{P}$ is a field, so P is maximal. □

Proposition 2 *Let R be Noetherian.*

1. *Some finite product of prime ideals of R is 0.*
2. *Given any ideal I of R, there exists a finite set $P_1, \ldots, P_t \supseteq I$ of prime ideals such that $P_1 \ldots P_t \subseteq I$.*

Proof. First note that by $I = 0$ (2) implies (1). Now we prove (2). Suppose (2) is false, and consider the set of counterexamples to (2). Since R is Noetherian this set has a maximal element, call it I. I is not prime, sine all prime ideals satisfy the condition of (2) using $t = 1$. So there exist ideals $A, B \supsetneq I, AB \subseteq I$, so A, B are not counterexamples to (2), so $\exists P_1, \ldots, P_t \supseteq A, P_1 \ldots P_t \subseteq A$ and $Q_1, \ldots, Q_s \supseteq B, Q_1 \ldots Q_s \subseteq B$, so

$$I \subseteq A \subseteq P_1, \ldots, P_t, \qquad I \subseteq B \subseteq Q_1, \ldots, Q_s,$$

but $P_1 \ldots P_t Q_1 \ldots Q_s \subseteq AB \subseteq I$, contradiction. □

Definition 1 *Let P be a prime ideal of R, and $S \subseteq R$. If there is no prime ideal Q with $S \subseteq Q \subsetneq P$, P is called minimal.*

Note. P is minimal over 0 iff *height* $P = 0$.

Proposition 3 *Let R be Noetherian, $S \subseteq R$. There are only finitely many prime ideals of R which are minimal over S.*

Proof. Let $I = \langle S \rangle$, then a prime ideal is minimal over S iff it is minimal over I. So by the previous proposition there exist $P_1, \ldots, P_t \supseteq I, P_1 \ldots P_t \subseteq I$. Take any prime ideal P which is minimal over I, so $P_1 \ldots P_t \subseteq P$ so some $P_i \subseteq P$. If $P_i \subsetneq P$, then $I \subseteq P_i \subsetneq P$, contradicting minimality, so $P_i = P$. □

Proposition 4 *R is Noetherian and $K - \dim R = 0 \iff R$ is Artinian.*

Proof. Last time we showed R Artinian $\Rightarrow R$ Noetherian. Earlier today we proved R is Artinian $\Rightarrow K - \dim R = 0$, so $\Leftarrow$ is done, now consider $\Rightarrow$. Suppose R is Noetherian and $K - \dim R = 0$. Take $P_1, \ldots, P_t$ prime ideals with $P_1 \ldots P_t = 0$. Build a composition series similarly to last time. Consider ${}^{P_{i+1}P_{i+2}\ldots P_t}/_{P_i \ldots P_t}$, which is a vector space over ${}^{R}/_{P_i}$ since $K - \dim R = 0$. So P_i is maximal, so by Noetherianness this vector space is finite dimensional. So put together the pieces as in the Artinian $\Rightarrow$ Noetherian proof. □

Example 1 *Suppose R is a PID, take $0 \neq r \in R$. ${}^{R}/_{Rr}$ has $K - \dim 0$, so is Artinian and Noetherian.*

Corollary 1 *Let R be Noetherian, let P be a height 0 prime ideal. Then R_P is Artinian.*

2 The Principal Ideal Theorem

Theorem 1 (Principal Ideal Theorem) *Let R be Noetherian, $P \in SpecR$. If P is minimal over $0 \neq a \in P$, then height $P \leq 1$.*

Proof. Suppose on the contrary that $P'' \subsetneq P' \subsetneq P$ prime ideals. Then ${}^{P}/_{P''}$ is minimal over the image of a in ${}^{R}/_{P''}$. So we can replace R with ${}^{R}/_{P''}$ and hence we can assume (1) R is an integral domain, and (2) $0 \subsetneq P' \subsetneq P$.

Now localize at P. P_P is a prime ideal of R_P and is minimal over $\frac{a}{1}$. So we can replace R with R_P and hence can assume (3) R is local with unique maximal ideal P. So now suppose all this. P is minimal over a since P is the unique maximal ideal, P is the only prime ideal of R containing a. In particular $a \notin P'$. Observe that P is minimal over any power of a, because if $a^n \in Q \subseteq P$ then $a \in Q$ so $Q = P$. Pick $0 \neq b \in P'$. Let $A_i = \{r \in R : ra^i \in Rb\}$. Then $A_i \subseteq A_{i+1}$. So since R is Noetherian, n exists such that $A_n = A_{n+1} = \ldots = A_{2n} = \ldots$, so if $ra^{2n} \in Rb$, then $ra^n \in Rb$. P is minimal over a^n, so replace a by a^n, and we have

$$ra^2 \in Rb \Rightarrow ra \in Rb. \qquad (*)$$

Let

$$M = Ra + Rb, \quad M' = Ra^2 + Rb.$$

Note M' is a submodule of M. We will get a contradiction by proving $M = M', M \neq M'$.

- First let's show $M \neq M'$: Suppose on the contrary $M = M'$, then $a = r_1a^2 + r_2b, r_1, r_2 \in R$. So $(1-r_1a)a = r_2b \in P'$, where $1-r_1a$ is invertible, so $a \in P'$, contradiction.

- Now let's show $M = M'$. Let $\bar{R} = {}^{R}/_{Ra^2}$, this is Noetherian and $\bar{P} = {}^{P}/_{Ra^2}$ is the only prime ideal, so $K - \dim \bar{R} = 0$. So $\bar{R}$ is Artinian, so $\bar{R}$ has a composition series. It suffices to prove ${}^{M}/_{Ra^2} = {}^{M'}/_{Ra^2}$. Both of these are finitely generated $\bar{R}-$modules, since they are finitely generated $R-$modules and are annihilated by Ra^2. So ${}^{M}/_{Ra^2}$ and ${}^{M'}/_{Ra^2}$ are both Artinian and Noetherian and so have finite composition length. Since one is contained in the other it suffices to show they have the same length. Let
$$N = Ra, \quad K = Ma = Ra^2 + Rba.$$

  Claim. $l({}^{M}/_{N}) = l({}^{K}/_{Ra^2})$.

  Proof of Claim. In any integral domain with $A \subsetneq B$ ideal, $r \neq 0, r \in R$, take
$$\begin{aligned} B &\to {}^{rB}/_{rA} \\ b &\mapsto rb + rA. \end{aligned}$$
  The kernel of this map is A, so by First Isomorphism Theorem ${}^{B}/_{A} \cong {}^{rB}/_{rA}$, so returning to our context ${}^{M}/_{N} \cong {}^{Ma}/_{Na} = {}^{K}/_{Ra^2}$. ■

  Claim. $l({}^{N}/_{Ra^2}) = l({}^{M'}/_{K})$.

  Proof of Claim. ${}^{N}/_{Ra^2} = {}^{Ra}/_{Ra^2} \cong {}^{R}/_{Ra}$. Let
$$\begin{aligned} f : &R \to {}^{M'}/_{K} \\ &r \mapsto rb + K. \end{aligned}$$
  Then $r \in \ker f \iff rb = r_1a^2 + r_2ba$ for some $r_1, r_2 \in R$. If $r \in Ra$ then $rb = r_2ab$, so $r \in \ker f$. If $r \in \ker f$ then $r_1a^2 = (r - r_2a)b \in Rb$. Then by $(*)$ $r_1a \in Rb$, so $r_1a = r_3b$ so $rb = r_2ab + r_3ab$, so $r = (r_2 + r_3)a \in Ra$. So $\ker f = Ra$, so ${}^{R}/_{Ra} \cong {}^{M'}/_{K}$ so ${}^{N}/_{Ra^2} \cong {}^{M'}/_{K}$, which proves the claim. ■

  The two claim above give
$$\begin{aligned} l({}^{M}/_{Ra^2}) &= l({}^{M}/_{N}) + l({}^{N}/_{Ra^2}) \\ &= l({}^{K}/_{Ra^2}) + l({}^{M'}/_{K}) \\ &= l({}^{M'}/_{Ra^2}), \end{aligned}$$
  so $M = M'$.

So we got contradiction, which completes the whole proof. □

Corollary 2 *Suppose R is Noetherian, and $P \supsetneq P_1 \supsetneq \ldots \supsetneq P_k$ are prime ideals of R and $0 \neq a \in P$, then $\exists P_i' \in SpecR$ such that $P \supsetneq P_1 \supsetneq \ldots \supsetneq P_{k-1}' \supsetneq P_k$, and $a \in P_{k-1}'$.*

Proof. We prove this by induction on k. As before replace R by ${}^{R}/_{P_k}$, so assume R is an integral domain and $P_k = 0$.

If $k = 2$ the statement says *height* $P > 1$. So by the Principal Ideal Theorem P is not minimal over a which implies the existence of P'_1.

For $k > 2$, by induction $\exists P'_1, \ldots, P'_{k-2} \in SpecR$ such that $P \supsetneq P'_1 \supsetneq \ldots \supsetneq P'_{k-2} \supsetneq P_{k-1} \supsetneq 0$, and $a \in P'_{k-2}$. Now apply $k = 2$ case to get P'_{k-1} with $P \supsetneq P'_1 \supsetneq \ldots \supsetneq P'_{k-2} \supsetneq P'_{k-1} \supsetneq 0$, and $a \in P'_{k-1}$. □

Theorem 2 (Generalized Principal Ideal Theorem) *Let R be a Noetherian ring, $P \in SpecR$. Let $A = \sum_{i=1}^{t} Ra_i$ be an ideal of R. If P is minimal over A then height $P \leq t$.*

Proof. We prove the theorem by induction on t. $t = 1$ case is just the Principal Ideal Theorem. Now assume $t > 1$: Suppose $P \supsetneq P_1 \supsetneq \ldots \supsetneq P_k$, we want to show that $k \leq t$. By the corollary above we can assume that $a_t \in P_{k-1}$ so ${}^{P}/_{Ra_t} \supsetneq {}^{P_1}/_{Ra_t} \supsetneq \ldots \supsetneq {}^{P_{k-1}}/_{Ra_t}$ is a chain of length $k-1$ in ${}^{R}/_{Ra_t}$, and ${}^{P}/_{Ra_t}$ is minimal over ${}^{A}/_{Ra_t} = \left(\sum_{i=1}^{t-1} Ra_i\right) + Ra_t$, so by induction $k - 1 \leq t - 1$, so $k \leq t$. □

Corollary 3 *Let R be Noetherian, $P \in SpecR$. If P is spanned by t elements then height $P \leq t$.*

Proof. Set $P = A$ in the previous result. □